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The torsional rigidity of anisotropic prisms

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SUMMARY
Upper and lower bounds are obtained for the torsional rigidity of a prismatic cylinder of non-homogeneous anisotropic elastic material. Improvement in the bounds is obtained by expressing each bound as the quotient of two bordered determinants. Some analytical and numerical results are also presented.

Introduction

The number of closed form solutions of the torsion problem for a non-homogeneous anisotropic medium is small. Some elementary solutions have been obtained by Chen [1], and by Brown and Jones [2] for curvilinearly aeolotropic material. The difficulties inherent in the problem have led to the search for methods which produce approximate solutions, and thus to estimates of the magnitude of the torsional rigidity of prismatic cylinders.

For material which is homogeneous and isotropic Prager [3] has evolved a method which provides upper and lower bounds to the torsional rigidity in terms of approximating functions derived from two basic energy extremum principles of elasticity theory. An alternative derivation has been given by Diaz [4], which also includes a method of improvement of the bounds.

For homogeneous orthotropic material Love [5] has shown that the torsional rigidity may be determined from that for an associated isotropic material by introducing a suitable coordinate transformation, and Lekhnitskii [6] has derived bounds for the torsional rigidity of cylinders of special cross-sections formed from orthotropic material. The method is also applicable to materials of more general anisotropy. Flavin [7] used the methods of [3, 5] to investigate further the bounds on the torsional rigidity of prisms of orthotropic material.

In this paper the prism is assumed to have a general cross-section, and is formed from non-homogeneous elastic material with a plane of elastic symmetry at each point perpendicular to the axis of the prism. A boundary-value problem approach is used to produce bounds for the torsional rigidity in terms of approximating functions following the method of Diaz [4] for isotropic material. A feature of the study is that the bounds are expressed in closed form for any degree of approximation using the Rayleigh–Ritz technique, and these form lend themselves readily to numerical evaluation.

1. The torsional rigidity

Let the right section of the prism be parallel to the x_1–x_2 plane, with the axis of the prism along the x_3 direction. If the twist per unit length due to applied end couples is θ then the displacement has components

$$u_1 = -\theta x_2 x_3\,,\quad u_2 = \theta x_1 x_3\,,\quad u_3 = \theta\phi(x_1, x_2)\,,$$

where $\phi(x_1, x_2)$ is the warping function. The corresponding stress-strain relations for a non-homogeneous material with a plane of elastic symmetry at each point normal to the axis of the prism are

$$\sigma_{13} = \theta\{\alpha_{11}(\phi_{,1} - x_2) + \alpha_{12}(\phi_{,2} + x_1)\}\,,$$

$$\sigma_{23} = \theta\{\alpha_{22}(\phi_{,2} + x_1) + \alpha_{21}(\phi_{,1} - x_2)\}\,,$$

or

$$\sigma_{i3} = \theta\alpha_{ij}\lambda_j \quad (i, j = 1, 2)\,, \tag{1.1}$$

where

$$\lambda_i = \phi_{,i} + e_{ji}x_j\,, \tag{1.2}$$

and

$$e_{12} = 1\,,\ e_{21} = -1\,,\ e_{11} = e_{22} = 0\,.$$

Here and subsequently a repeated index implies summation over all the values of that index, and $({}_{,i})$ implies differentiation with respect to x_i. For a non-homogeneous material the α_{ij} are functions of x_1 and x_2, and as usual $\alpha_{12} = \alpha_{21}$.

The equilibrium equations in this case are

$$\sigma_{i3,i} = 0\,,$$

hence from (1.1) we have

$$(\alpha_{ij}\lambda_j)_{,i} = 0\,. \tag{1.3}$$

The force on the lateral surface of the prism is zero, hence if n_i is the unit outward drawn normal along C, the perimeter of a right-section, then

$$n_i\sigma_{i3} = 0\,,$$

and thus from (1.1) the boundary condition on C is

$$n_i\alpha_{ij}\lambda_j = 0\,. \tag{1.4}$$

The torsional rigidity T of the prism is defined as the moment of the couple required to produce unit twist, and is given by

$$T = \frac{1}{\theta}\int_S (\sigma_{23}x_1 - \sigma_{13}x_2)\mathrm{d}S\,,$$

where S is the area of the right-section. This may be written in the form

$$T = \frac{1}{\theta}\int_S e_{ji}\sigma_{i3}x_j\mathrm{d}S\,,$$

and from (1.1) and (1.2) we have

$$T = I - P\,, \tag{1.5}$$

where

$$P = -\int_S e_{ji}\alpha_{ik}\phi_{,k}x_j\mathrm{d}S\,,$$

and

$$I = \int_S e_{ji}e_{mk}\alpha_{ik}x_m x_j\mathrm{d}S$$

$$= \int_S (\alpha_{11}x_2^2 + \alpha_{22}x_1^2 - 2\alpha_{12}x_1x_2)\mathrm{d}S\,. \tag{1.6}$$

Now

$$(\alpha_{ij}\phi\lambda_j)_{,i} = \alpha_{ij}\phi_{,i}\lambda_j + \phi(\alpha_{ij}\lambda_j)_{,i}$$
$$= \alpha_{ij}\phi_{,i}(\phi_{,j} + e_{kj}x_k)$$

by use of (1.2) and (1.3). Hence on rearrangement

$$(\alpha_{ij}\phi\lambda_j)_{,i} = \alpha_{ij}\phi_{,i}\phi_{,j} + e_{ji}\alpha_{ki}\phi_{,k}x_j\,,$$

and thus P reduces to

$$P = \int_S \alpha_{ij}\phi_{,i}\phi_{,j}dS - \int_S (\phi\alpha_{ij}\lambda_j)_{,i}dS .$$

By use of Gauss' theorem the second integral becomes

$$\int_C \phi\alpha_{ij}\lambda_j n_i ds ,$$

which is zero in value from (1.4), and then

$$P = \int_S \alpha_{ij}\phi_{,i}\phi_{,j}dS . \tag{1.7}$$

That P is a positive definite quadratic form follows from the fact that the strain energy function

$$\begin{aligned} W &= \tfrac{1}{2}(\alpha_{22}\varepsilon_{23}^2 + 2\alpha_{12}\varepsilon_{23}\varepsilon_{13} + \alpha_{11}\varepsilon_{13}^2) \\ &= \tfrac{1}{2}\alpha_{ij}\varepsilon_{i3}\varepsilon_{j3} > 0 , \end{aligned} \tag{1.8}$$

and a necessary and sufficient condition for this to be so is that

$$\alpha_{11} > 0 , \quad \alpha_{22} > 0 , \quad \alpha_{11}\alpha_{22} - \alpha_{12}^2 > 0 . \tag{1.9}$$

It follows that $P > 0$, and thus from (1.5) that

$$T < I , \tag{1.10}$$

giving a rough upper bound for the torsional rigidity.

2. Upper bound for the torsional rigidity

In (1.7) we write $\phi = \chi + (\phi - \chi)$, where χ is as yet an arbitrary function of x_1 and x_2 in $S + C$. It follows that

$$P \geqq \int_S \alpha_{ij}\chi_{,i}\chi_{,j}dS + 2\int_S \alpha_{ij}\chi_{,i}(\phi - \chi)_{,j}dS ,$$

since α_{ij} is symmetric and the components satisfy (1.9), or

$$P \geqq 2\int_S \alpha_{ij}\phi_{,j}\chi_{,i}dS - \int_S \alpha_{ij}\chi_{,i}\chi_{,j}dS .$$

But from (1.2) we have

$$\int_S \alpha_{ij}\phi_{,j}\chi_{,i}dS = \int_S \alpha_{ij}\lambda_i\chi_{,j}dS - \int_S \alpha_{ij}e_{ki}x_k\chi_{,j}dS .$$

The first integral on the right-hand side of this equation may be written as

$$\int_S (\alpha_{ij}\lambda_i\chi)_{,j}dS - \int_S \chi(\alpha_{ij}\lambda_i)_{,j}dS ,$$

and this complete expression is zero by use of Gauss' theorem together with (1.3) and (1.4). In all it follows that

$$T \leqq I + \int_S \alpha_{ij}\chi_{,i}\chi_{,j}dS - 2\int_S \alpha_{ij}e_{ik}x_k\chi_{,j}dS , \tag{2.1}$$

and by comparison with (1.8) the first integral on the right-hand side is positive definite.

In the application of Gauss' theorem it is sufficient that χ be continuous in S, and this is the only restriction on χ in S.

3. Lower bound for the torsional rigidity

Let μ_i be an arbitrary function of x_1 and x_2 in S, then writing as before

$$\phi_{,i} = \mu_i + (\phi_{,i} - \mu_i),$$

we have from (1.7)

$$P \geqq 2\int_S \alpha_{ij}\phi_{,j}\mu_i dS - \int_S \alpha_{ij}\mu_i\mu_j dS. \quad (3.1)$$

Consider next the integral

$$R = \int_S \alpha_{ij}\phi_{,j}(\mu_i - \phi_{,i})dS$$
$$= \int_S \{\alpha_{ij}\phi(\mu_i - \phi_{,i})\}_{,j}dS - \int_S \phi\{\alpha_{ij}(\mu_i - \phi_{,i})\}_{,j}dS,$$

and let

$$\alpha_{ij}(\mu_i - \phi_{,i})_{,j} = 0 \quad (3.2)$$

in S. Then by Gauss' theorem

$$R = \int_C \alpha_{ij}\phi(\mu_i - \phi_{,i})n_j ds,$$

and it is noted that $\alpha_{ij}\mu_i n_j$ must be continuous in S.

Again assume that

$$\alpha_{ij}(\mu_i - \phi_{,i})n_j = 0 \quad (3.3)$$

on C, then $R=0$, or

$$\int_S \alpha_{ij}\phi_{,j}\mu_i dS = \int_S \alpha_{ij}\phi_{,j}\phi_{,i}dS = P.$$

It follows that (3.1) may be rewritten as

$$P \leqq \int_S \alpha_{ij}\mu_i\mu_j dS,$$

and thus

$$T \geqq I - \int_S \alpha_{ij}\mu_i\mu_j dS. \quad (3.4)$$

If we now introduce another function given by

$$\rho_i = \mu_i + e_{ji}x_j,$$

then (3.2), together with (1.2) and (1.3), reduces to

$$\alpha_{ij}\rho_{i,j} = 0 \quad (3.5)$$

in S. It is noted that $\alpha_{ij}\rho_i n_j$ must be continuous in S.

Again (3.3), with the use of (1.4), reduces to

$$\alpha_{ij}\rho_i n_j = 0 \quad (3.6)$$

on C. In terms of ρ_i the lower bound for the torsional rigidity (3.4) takes the form

$$T \geqq 2\int_S \alpha_{ij}e_{ki}x_k\rho_j dS - \int_S \alpha_{ij}\rho_i\rho_j dS, \quad (3.7)$$

and ρ_i satisfies (3.5) and (3.6).

Choose a function ψ, which is such that

$$\alpha_{ij}\rho_j = e_{ij}\psi_{,j}, \tag{3.8}$$

then (3.5) is satisfied identically, since $e_{ij}\psi_{,ij}=0$. Again (3.6) becomes

$$e_{ij}\psi_{,j}n_i = 0$$

on C. If t_i is the unit vector tangential to C then $t_i = e_{ki}n_k$, and thus $t_j\psi_{,j}=0$, or $d\psi/ds=0$ along C, implying that

$$\psi = \text{constant} \tag{3.9}$$

on C. Also $e_{ij}\psi_{,j}n_i$ must be continuous in S, i.e. ψ must be continuous in S.

The lower bound (3.7) for T may also be written in terms of ψ. Thus let β_{ij} be such that

$$\beta_{ji}\alpha_{ik} = \delta_{jk}, \tag{3.10}$$

so that from (3.8) we have

$$\rho_i = \beta_{ji}e_{ik}\psi_{,k}.$$

In this case (3.10) may be solved for β_{ij} leading to

$$\beta_{11} = K\alpha_{22}, \quad \beta_{22} = K\alpha_{11}, \quad \beta_{12} = \beta_{21} = -K\alpha_{12},$$

where $K = 1/(\alpha_{11}\alpha_{22} - \alpha_{12}^2)$, and then it readily follows that (3.7) reduces to

$$T \geqq -2\int_S x_1\psi_{,i}\,dS - K\int_S \alpha_{ij}\psi_{,i}\psi_{,j}\,dS, \tag{3.11}$$

which provides a lower bound to the torsional rigidity in terms of a function ψ, which is continuous in S and constant on C. Again by comparison with (1.8) the second integral on the right-hand side is positive definite.

4. Determination and improvement of the bounds

4.1. *Upper bound for the torsional rigidity*

The upper bound for T is given by (2.1) as $T \leqq U$, where U is given by the right-hand side of (2.1), and here χ is continuous in S, but otherwise arbitrary.

Since U is a scalar when χ is chosen to be scalar, then the upper bound to T will be independent of the orientation of the coordinate axes. Following the standard Rayleigh–Ritz technique [8, 9] we thus express χ in the form

$$\chi = \omega + \omega_r f_r + \omega_{rs} f_{rs} + \omega_{rst} f_{rst} + \dots,$$

where $\omega_{rs\dots}$ are arbitrary constant tensors, and the $f_{rs\dots}$ are prescribed tensor functions of the coordinates. It is however more convenient to express χ in the form

$$\chi = \sum_{r=0}^{n} a_r g_r(x_1, x_2), \tag{4.1}$$

where a_r and g_r are related to $\omega_{rs\dots}$ and $f_{rs\dots}$ respectively.

The expression (4.1) for χ may be substituted into U, the right-hand side of (2.1), and the resulting equation may be written as

$$U = I - 2\boldsymbol{a}\boldsymbol{Y} + \boldsymbol{a}\boldsymbol{X}\boldsymbol{a}^T, \tag{4.2}$$

where $\boldsymbol{a}$ is a row vector of order n with arbitrary constant elements a_r, $\boldsymbol{a}^T$ is its conjugate, $\boldsymbol{Y}$ is a column vector of order n with elements

$$Y_r = \int_S \alpha_{ij}e_{ik}x_k g_{r,j}\,dS, \tag{4.3}$$

and X is a positive definite square matrix of order n with elements

$$X_{rs} = \int_S \alpha_{ij} g_{r,i} g_{s,j} dS . \tag{4.4}$$

The minimum upper bound occurs when $\partial U/\partial a_r = 0$, and thus from (4.2) we have

$$Y = Xa^T \quad \text{or} \quad a = Y^T X^{-1} .$$

In this case (4.2) reduces to

$$U = I - Y^T X^{-1} Y . \tag{4.5}$$

Alternatively [10] we note that

$$U = \begin{vmatrix} X & Y \\ Y^T & I \end{vmatrix} \Big/ |X| , \tag{4.6}$$

and thus the least upper bound has been expressed as the ratio of two determinants, the numerator being a bordered form of that of the denominator. All the elements are known functions, and the form lends itself readily to computation. From (4.4) we have

$$X_{rs} = \int_S \{\alpha_{11} g_{r,1} g_{s,1} + \alpha_{12} (g_{r,1} g_{s,2} + g_{r,2} g_{s,1}) + \alpha_{22} g_{r,2} g_{s,2}\} dS, \tag{4.7}$$

and from (4.3)

$$Y_r = \int_S \{\alpha_{11} x_2 g_{r,1} + \alpha_{12} (x_2 g_{r,2} - x_1 g_{r,1}) - \alpha_{22} x_1 g_{r,2}\} dS , \tag{4.8}$$

and I is given by (1.6).

4.2. *Lower bound for the torsional rigidity*

The lower bound L is defined by the right-hand side of (3.11), and ψ is continuous in S, and is constant on C. In the same manner as for the upper bound a series of functions is chosen in the form

$$\psi = \sum_{r=0}^{n} b_r h_r (x_1, x_2) , \tag{4.9}$$

where the b_r are arbitrary constants, and the h_r are prescribed functions continuous in S, and chosen to make ψ constant on C. This expression is substituted into L, the right-hand side of (3.11), leading to

$$L = -2bZ - KbWb^T , \tag{4.10}$$

where b is a row vector of order n with elements b_r, Z is a column vector of order n with elements

$$Z_r = \int_S x_i h_{r,i} dS , \tag{4.11}$$

and W is a symmetric positive definite square matrix of order n with elements

$$W_{rs} = \int_S \alpha_{ij} h_{r,i} h_{s,j} dS . \tag{4.12}$$

The maximum lower bound is derived from $\partial L/\partial b_r = 0$, and then from (4.10) we have

$$Z = -KWb^T \quad \text{or} \quad b = -K^{-1} Z^T W^{-1} .$$

In this case (4.10) reduces to

$$L = K^{-1} Z^T W^{-1} Z ,$$

or alternatively

$$L = -\begin{vmatrix} W & Z \\ Z^T & 0 \end{vmatrix} \Big/ K|W| . \tag{4.13}$$

This form is again the ratio of two determinants, one a bordered form of the other, the elements of which are known, since from (4.12) and (4.11) respectively

$$W_{rs} = \int_S \{\alpha_{11} h_{r,1} h_{s,1} + \alpha_{12}(h_{r,1} h_{s,2} + h_{r,2} h_{s,1}) + \alpha_{22} h_{r,2} h_{s,2}\} dS , \tag{4.14}$$

and

$$Z_r = \int_S (x_1 h_{r,1} + x_2 h_{r,2}) dS . \tag{4.15}$$

5. Analytical results

The bounds given by (4.6) and (4.13) are of the same type, both being the ratio of two determinants, the numerator being a bordered version of the denominator. Such ratios may be expanded as a Schweinsian expansion [11] for given n. For convenience consider the upper bound U defined by (4.6). When n functions g_r are included in the series (4.1) then the corresponding value of U can be referred to as U_n, and it follows from [11] that

$$U_n - U_{n-1} = -(|Y_1 X_{12} X_{23} \dots X_{n-1\,n}|)^2 / X_{n-1} X_n , \tag{5.1}$$

where the Gram determinant $X_n \equiv |X_{11} X_{22} \dots X_{nn}|$. The determinants are referred to by the elements in their leading diagonals. It follows immediately that if the number of prescribed functions in the approximating series (4.1) is increased by one, then the change in the value of the determinantal ratio is given by (5.1). When the number of functions in (4.1) changes from m to n then by repeated application of formulae similar to (5.1) it is possible to obtain a result for $U_n - U_m$.

The quadratic form $\boldsymbol{aXa}^T$ is positive definite, and the condition for this to be so is that the Gram determinants X_r, $(r=1, 2, \dots, n)$, are positive, hence the right-hand side of (5.1) is always negative, and thus an increase in the number of approximating functions in (4.1) will decrease the value of the upper bound, thus improving its value.

A repeated application of (5.1) and a summation produces the finite series expansion

$$U_n = I - \frac{Y_1^2}{X_1} - \sum_{r=2}^{n} \frac{(|Y_1 X_{12} X_{23} \dots X_{r-1\,r}|)^2}{X_{r-1} X_r} . \tag{5.2}$$

It is thus possible to find an exact formula for the least upper bound, for a given n.

The corresponding result for L_n derived from (4.13) is

$$L_n = K \left\{ \frac{Z_1^2}{W_1} + \sum_{r=2}^{n} \frac{(|Z_1 W_{12} W_{23} \dots W_{r-1\,r}|)^2}{W_{r-1} W_r} \right\} , \tag{5.3}$$

and since in (4.10) the quadratic form $\boldsymbol{bWb}^T$ is positive definite then $W_r > 0$, $(r=1, 2, \dots, n)$. Again it is noted that the series has a finite number of terms and $K > 0$. Hence in general an increase in n produces an increase in L_n, and thus produces an improved value of the lower bound. Results related to (5.2) and (5.3) for a different problem are quoted by Levine and Schwinger [12] without proof.

Example 1. *An upper bound for a homogeneous anisotropic elastic prism.*

The approximating function χ is given by (4.1) with $n=5$, and

$$g_1 = x_1 , \; g_2 = x_2 , \; g_3 = x_1 x_2 , \; g_4 = \tfrac{1}{2} x_2^2 , \; g_5 = \tfrac{1}{2} x_1^2 .$$

The corresponding functions X_{rs}, Y_r, $(r=1, 2, \dots, 5)$, from (4.7) and (4.8) are readily determined in terms of the integrals

$$I_{rs} = \int_S x_1^r x_2^s \, dS ,$$

it being remembered that for homogeneous material α_{ij}, $(i, j=1, 2)$, are constants. Row or column reduction of the determinants in (4.6) leads ultimately to the result

$$U = \frac{4(\alpha_{11}\alpha_{22}-\alpha_{12}^2)(\bar{I}_{02}\bar{I}_{20}-\bar{I}_{11}^2)}{\alpha_{11}\bar{I}_{02}+\alpha_{22}\bar{I}_{20}-2\alpha_{12}\bar{I}_{11}} , \tag{5.4}$$

where

$$\bar{I}_{rs} = \int_S (x_1-\bar{x}_1)^r (x_2-\bar{x}_2)^s \, dS .$$

Here $\bar{x}_1$, $\bar{x}_2$ are the coordinates of the centre of area of the cross-section S, $\bar{I}_{02}$ and $\bar{I}_{20}$ are the second moments of S about lines through the centre of area parallel to the x_1 and x_2 axes respectively, and $\bar{I}_{11}$ is the corresponding product moment.

The expression for U has the property that it is invariant under a rotation of the axial system. It is also noticed that it is an extension of the result given by Flavin [7, p. 702] for the orthotropic cylinder.

Example 2. *An upper bound for a non-homogeneous anisotropic elastic prism.*

It is known that a rough upper bound U for the torsional rigidity T is I, as shown in (1.10), where I is defined in (1.6), and the elastic parameters α_{ij} are functions of position. An improvement of the bound may be obtained by taking $n=2$ in (4.1) with $g_1=x_1$, $g_2=x_2$. In this case (5.2) becomes

$$U = I - \frac{X_{22}Y_1^2+X_{11}Y_2^2-2X_{12}Y_1Y_2}{X_{11}X_{22}-X_{12}^2} , \tag{5.5}$$

where

$$X_{rs} = \int_S \alpha_{rs} dS , \qquad (r, s=1, 2) ,$$

and

$$Y_r = \int_S (\alpha_{r1}x_2-\alpha_{r2}x_1) dS , \qquad (r=1, 2) .$$

This result (5.5) obtained by the inclusion of linear terms in x_1 and x_2 in the approximating function is precisely the smallest value of I, as defined in (1.6), obtained by a change in position of the origin of coordinates.

Example 3. *An upper bound for a non-homogeneous anisotropic elastic prism of symmetrical section.*

Here α_{ij} is a function of position, and the material non-homogeneity is such that it is symmetrical about the x_1 axis, coincident with the axis of symmetry of the prism section, i.e. $\alpha_{ij}(x_1, x_2)=\alpha_{ij}(x_1, -x_2)$. It is found convenient to introduce the integrals

$$I_{rs}^{ij} = \int_S \alpha_{ij}(x_1, x_2) x_1^r x_2^s \, dS ,$$

so that $I_{rs}^{ij}=0$ when s is an odd integer.

An approaching function χ in this case is chosen involving the functions

$$g_1 = x_1x_2 , \quad g_2 = \tfrac{1}{2}x_1^2 , \quad g_3 = \tfrac{1}{2}x_2^2 .$$

Use of (4.6) leads to the upper bound of the form

$$U = 4AB/(A+B) , \tag{5.6}$$

where

$$A = I_{02}^{11}-(I_{02}^{12})^2/I_{02}^{22} ,$$
$$B = I_{20}^{22}-(I_{20}^{12})^2/I_{20}^{11} .$$

It is noted that (5.6) has the same form as that given by Nicolai [13] for the isotropic prism.

Example 4. *Bounds for the homogeneous anisotropic N-fold symmetric elastic prism.*

It is known that the functions $g_r(x_1, x_2)$ of (4.1) are continuous in S. The prism with N-fold symmetry has N identical sections, and thus we choose the $g_r(x_1, x_2)$ to be identical in each section and continuous across the internal boundaries separating each section. In this context each section can be considered as a typical section S_0 rotated through an integral multiple of $2\pi/N$, and the upper bound formula (4.6) reduces to

$$U = N(\alpha_{11}+\alpha_{22}) \begin{vmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & J \end{vmatrix} \Big/ 2|\boldsymbol{A}|\,, \tag{5.7}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices with elements

$$A_{rs} = \int_{S_0} (g_{r,1}\, g_{s,1} + g_{r,2}\, g_{s,2})\, dS_0\,,$$

$$B_r = \int_{S_0} (x_2\, g_{r,1} - x_1\, g_{r,2})\, dS_0\,,$$

respectively, and

$$J = \int_{S_0} (x_1^2 + x_2^2)\, dS_0\,.$$

The lower bound is determined from (4.13) assuming that ψ can be developed as in (4.9) in each of the N parts of S, and $h_r(x_1, x_2)$ must be continuous across internal boundaries to each section of S. This bound reduces to

$$L = -2N(\alpha_{11}\alpha_{22} - \alpha_{12}^2) \begin{vmatrix} \boldsymbol{C} & \boldsymbol{D} \\ \boldsymbol{D}^T & 0 \end{vmatrix} \Big/ (\alpha_{11}+\alpha_{22})|\boldsymbol{C}|\,, \tag{5.8}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are matrices with elements

$$C_{rs} = \int_{S_0} (h_{r,1}\, h_{s,1} + h_{r,2}\, h_{s,2})\, dS_0\,,$$

$$D_r = \int_{S_0} (x_1\, h_{r,1} + x_2\, h_{r,2})\, dS_0\,,$$

respectively.

If U_I and L_I are the bounds for the corresponding isotropic prism of rigidity modulus μ, then

$$U = \frac{(\alpha_{11}+\alpha_{22})\, U_I}{2\mu}\,, \qquad L = \frac{2(\alpha_{11}\alpha_{22} - \alpha_{12}^2)\, L_I}{\mu(\alpha_{11}+\alpha_{22})}\,. \tag{5.9}$$

A knowledge of the bounds on the torsional rigidity for the isotropic prism thus leads to values of the bounds for an anisotropic prism.

Further, if the exact value T_I of the torsional rigidity for the isotropic prism is known, then bounds for the anisotropic prism can be found using U and L as in (5.9), with U_I and L_I replaced by T_I. This is a generalization of a result due to Flavin [7, pp. 700, 701]. The coefficients of U_I and L_I in (5.9) are scalars, and thus the bounds are independent of the orientation of the axes.

The regular hexagon section. For a prism of isotropic material of rigidity modulus μ, with a cross-section in the form of a regular hexagon of side a the torsional rigidity T_I has the (rounded-off) value $1.035459\ \mu a^4$. This result has been obtained by the application of a formula due to Seth [14]. It differs from the value given by Polya and Szego [15, p. 258], although it is consistent with results derived in Section 6 of this paper. If we write $P = \alpha_{11}/\alpha_{22}$, $Q = \alpha_{12}/\alpha_{22}$, $L' = L/\alpha_{22}a^4$, $U' = U/\alpha_{22}a^4$, then application of (5.9) with U_I and L_I replaced by T_I leads to the results as given in Table 1.

The upper bound suffers from the defect that it does not reflect any change in α_{12}. It should also be noted that these bounds can only lead to rough estimates of the values of the torsional rigidities, since the bounds differ by about 12%, 3% and 7% respectively from their mean values.

TABLE 1
Regular hexagon section

P	Q	L'	U'
0.50	0.25	0.604018	0.776594
1.00	0.25	0.970743	1.035459
2.00	0.25	1.337468	1.553188

6. Numerical results

The prism is assumed to be elastically homogeneous and anisotropic, so that α_{ij}, $(i, j=1, 2)$, are constants. The method applies equally to non-homogeneous prisms. It is assumed that the function χ of (4.1) is developed as a sum of polynomials of degrees ranging from zero to a selected integral value n, with arbitrary constant coefficients a_r. It is thus possible to write

$$\chi = \sum_{p=0}^{n} \sum_{q=0}^{p} a_r x_1^{p-q} x_2^q \,, \tag{6.1}$$

where $r=\frac{1}{2}p(p+1)+q$, and for a given integer n the total number of terms in χ is $m=\frac{1}{2}(n+1)(n+2)$.

In the case of ψ, given by (4.9), it is necessary to write

$$\psi = f(x_1, x_2) \sum_{p=0}^{n} \sum_{q=0}^{p} b_r x_1^{p-q} x_2^q \,, \tag{6.2}$$

where $f(x_1, x_2)$ is a polynomial which is zero on C the contour of the prism cross-section.

The matrix elements X_{rs}, Y_r of (4.7), (4.8) respectively involve integrals of the type

$$\int_S x_1^\alpha x_2^\beta dx_1 dx_2 \,,$$

where α, β are integers. These integrals may be evaluated directly for an area S with contour C, or by an iterative method. Due note should be made of any geometrical symmetries in the cross-section, so that some of the integrals may be zero in value, and thus some of the matrix elements may have zero values, or equivalently some of the terms in χ or ψ may be omitted.

If χ contains m terms then the determinant in the numerator of U of (4.6) is of order $m+1$, the leading minor of order m being identical with the determinant in the denominator of (4.6). In practice the positive definiteness of both determinants allows Gaussian elimination to proceed without pivoting, so that both determinants can be reduced to upper triangular form simultaneously. The ratio of the original determinants will then have a value equal to that of the $(m+1)$th element in the leading diagonal of the upper triangular determinant in the final form of the numerator. The error produced in this element due to the elimination process is given by Kabaza [16] as less than $2.01mae$, where a is the value of the largest element in the original numerator determinant, and e is the least significant digit in the word of the binary digital computer in use ($e=10^{-16}$ in this computation). The computer program was arranged so as to obtain the values of the ratio of the determinants at intermediate values of the range $(0, m)$ during the same elimination process. In order to counteract the effect of possible instability due to the choice of approximating function the numerical analysis was repeated using double-length arithmetic, and a further check of the accuracy was made by using the method of reliable bounds following Synge [17]. To the number of decimal places given in the following numerical results the values were the same. The method may be applied equally well to the evaluation of L from (4.13).

Let $P=\alpha_{11}/\alpha_{22}$, $Q=\alpha_{12}/\alpha_{22}$, and $U'=U/\alpha_{22}a^4$, $L'=L/\alpha_{22}a^4$, where a is a standard length associated with the cross-section of the prism.

6.1. *Kite section*

In Fig. 1 the line BD is orthogonal to AC, and $AO = OC = a/2$, $BE = ED$, $M = BD/AC$, $R = 2OE/AC$. Table 2 gives the values for the bounds L' and U' for varying values of P and M, with $n = 10$ in (6.1), (6.2) respectively, and $Q = 0.25$, $R = 0.5$.

For isotropic material with $P = 1$, $Q = 0$ the results for the square section, i.e. $M = 1$, $R = 0$ are

$$L' = 0.0351\,4415\,, \quad U' = 0.0351\,4487\,,$$

the exact (rounded-off) value being 0.0351 4428.

For the rhombus section with $M = 3^{-\frac{1}{2}}$, $R = 0$ we have

$$L' = 0.0104\,3670\,, \quad U' = 0.0104\,3967\,.$$

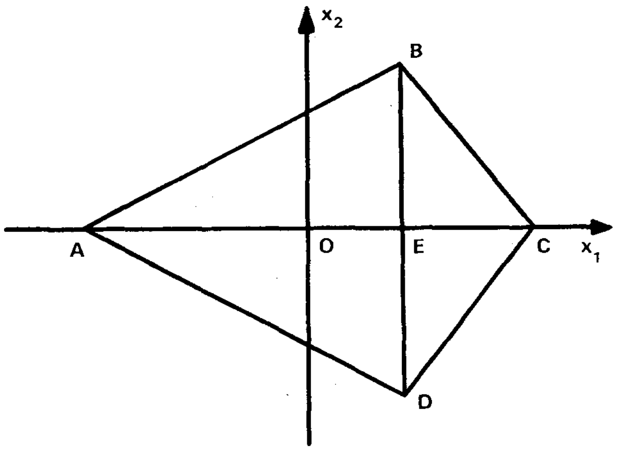

Figure 1. Kite section.

TABLE 2
Kite section

P	M	L'	U'
0.5	1.5	0.0035 8042	0.0035 8378
0.5	1.0	0.0198 4654	0.0198 5016
0.5	0.5	0.0469 0588	0.0469 2572
2.0	1.5	0.0109 4203	0.0109 4369
2.0	1.0	0.0441 5723	0.0441 8884
2.0	0.5	0.0853 8937	0.0854 9254

TABLE 3
Isosceles triangle section

P	M	L'	U'
0.5	0.5	0.0030 2660	0.0030 2674
0.5	1.0	0.0168 3210	0.0168 3218
0.5	1.5	0.0409 5216	0.0409 5219
2.0	0.5	0.0092 0390	0.0092 0392
2.0	1.0	0.0392 3355	0.0392 3356
2.0	1.5	0.0790 5765	0.0790 5821

TABLE 4
Right-angled triangle section

P	M	L'	U'
0.5	0.5	0.0027 1111	0.0027 1167
0.5	1.0	0.0139 7942	0.0139 8037
2.0	0.5	0.0081 2412	0.0081 2431
2.0	1.0	0.0324 9649	0.0324 9724

TABLE 5
Regular hexagon section

P	Q	L'	U'
0.5	0.25	0.6067 9483	0.6069 3118
2.0	0.25	1.3413 3400	1.3415 7706

6.2. *Isosceles triangle section*

For an isosceles triangle section ABC of base $AB = Ma$, and height a, the bounds for $n = 10$, $Q = 0.25$ are recorded in Table 3 for varying P and M.

For an equilateral triangle section of a prism of isotropic material both bounds have the same value 0.0384 9002, the exact (rounded-off) value, as is to be expected with a polynomial approximating function.

6.3. *Right-angled triangle section*

The triangle ABC has a right angle at B, and $AB = a$, $BC = Ma$. For $Q = 0.25$ and $n = 10$ the bounds for the torsional rigidity are given by Table 4.

For a prism of isotropic material of 30°–60°–90° triangle section, i.e. $P = 1$, $Q = 0$, $M = 3^{-\frac{1}{2}}$, the bounds are

$$L' = 0.0079\,1405\,, \quad U' = 0.0079\,1411\,,$$

and these may be compared with the exact (rounded-off) value 0.0079141 as given by Hay [18].

6.4. *Regular hexagon section*

For a hexagonal section of side a the lower and upper bounds to the torsional rigidity are given in Table 5. Symmetry properties of the matrices $\boldsymbol{X}$ and $\boldsymbol{Z}$ in (4.6) and (4.13) respectively have allowed the values of $n = 16$ to be taken for L' and U' without use of excessive store space during the computation.

The results for the case of isotropic material where $P = 1$, $Q = 0$, are

$$L' = 1.0354\,1891\,, \quad U' = 1.0355\,4020\,,$$

and this should be compared with the exact (rounded-off) result obtained by the method of Seth [14], i.e. 1.035459, as referred to in Section 5. The bounds in Table 5 are an improvement on the corresponding results of Table 1.

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